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A BAYESIAN APPROACH TO SEASONAL STYLE GOODS FORECASTING

A THESIS

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SUMMARY

Seasonal style goods present special problems to managers because of their finite period of marketability and their seasonal demand pattern. Assuming a season consists of n periods, management tries to satisfy demand on a period by period basis rather than on a seasonal basis. Over ordering a style within a season to fulfill customer demand may result in a surplus at the end of the season. Since a style is, in general, fashionable for only one season, surplus units must be marked down, resulting in a smaller profit or a loss.

The need for an accurate forecasting technique in a seasonal style goods environment is apparent. Market surveys, sales managers' advice, historical data on style lines, and similar information help management estimate the demand for a given style initially. However, management needs a formal procedure to blend the knowledge with period by period demand observations to form revised estimates of future periods' demands.

Bayesian probability concepts are examined for their applicability to the seasonal style goods forecasting problem. Forecasting models are developed for two cases. In the first case, the fraction of total demand to be realized in each period of the season is assumed to be known, but the total demand is unknown. A general solution is presented. In the second case, neither the fractions of total demand for the periods nor the total demand is known. A restricted solution is offered that requires management to estimate two of four unknown model parameters so that revisions of future periods' demands may be made.

In practice, management can often provide estimates of the required parameters based on prior knowledge, reducing the severity of the restriction.

CHAPTER I

INTRODUCTION

The Problem

Demand forecasting is essential for planning in both industrial and commercial environments. Manufacturers must make timely decisions with regard to production and work force levels, budgetary needs, and facilities' requirements. Similarly, retailers have to determine product line composition and inventory levels. Well designed forecasting procedures provide valuable information that will assist management in making the most profitable choice among decision alternatives.

Forecasting models are developed for particular situations and under various assumptions. Of present concern are seasonal style goods, which are characterized by their finite period of marketability as well as by their seasonal demand pattern. In practice, styles are classified by seasonal patterns, and other qualities, into product lines. The styles composing a line change between seasons, and a new style may require the development of a new line. A forecasting model for these items cannot be based entirely on historical data for two reasons: (1) if a new style cannot be classified into an existing style line, past data will not exist, and (2) market research projections, sales manager's predictions and similar information represent a current feel for the upcoming season that should be incorporated formally into the model. Also, the forecasting procedure should be able to effectively utilize demand observations within the season. Therefore, a well designed forecasting model will consider subjectively obtained data (e.g., market surveys) and objective data (i.e., historical data and observations on the current demand process).

In the present study, the problem is to develop a seasonal style goods forecasting model that will blend data obtained from both subjective and objective sources into accurate and reliable forecasts of demand.

For convenience, the terms "products" and "goods" will henceforth be interpreted as seasonal style goods, and similarly, the term "subjective data" will mean subjectively obtained data or information.

Background

Because of their inherent nature, seasonal style goods present difficulties not encountered with continuously demanded items. The following is a brief analysis of the more common problem areas.

As a season is of finite length, two conflicting costs must be considered, opportunity costs and obsolescence costs. Opportunity costs represent the costs associated with unfulfilled demand such as unrealized profit and loss of goodwill. Obsolescence costs are incurred when a surplus of products remains at the end of the season. These goods must be disposed of at a reduced price, resulting in a smaller profit or a loss. Even when a style will remain popular for another season, prices at the end of the season are still marked down to avoid the costs of storage and deterioration between seasons. Although obsolescence costs are associated with the end of a season, these costs are caused by over ordering during the season to avoid opportunity costs.

A season is assumed to consist of n periods in which demand may be observed. The characteristic seasonal pattern is the result of fluctuating demand rates from period to period. Because of the varying demand rates, ordering the correct amount of product for future periods is very difficult without accurate forecasts.

As noted earlier, historical data is not available for a new style that requires the development of a new line. In this case, management must rely on purely subjective data as a basis for decisions prior to the beginning of the season.

Because of the short duration of a season, there are only a few opportunities to produce or purchase a style. To minimize production costs, manufacturers have to allocate capacity optimally among items competing for production facilities; an additional, unexpected run of a style may result in not meeting other product commitments, in sacrificing timely entry into the market of another style good, or in incurring overtime costs. Retailers must place reorders within the first few periods of the season in order to receive delivery by a reasonable date.

Obsolescence and opportunity costs, fluctuating demand rates, lack of historical data, and few decision opportunities are some of the problems that make management of seasonal style goods difficult. Decision making within this environment is highly dependent upon the effectiveness of demand forecasting. If accurate forecasts are available early in the season, manufacturers can readjust production schedules to match demand without fear of incurring excessive obsolescence costs. Further, retailers are better able to make decisions regarding reorders or possible order cancellations. Therefore, a well designed forecasting procedure can reduce costs considerably.

Purpose of the Research

The purpose of the present study is to develop a forecasting model for seasonal style goods that can efficiently draw upon both subjective and objective information, yielding management accurate and reliable forecasts.

The problem will be approached through well known Bayesian probability concepts that are renowned for their ability to blend prior knowledge (subjective and historical data) with observations of a demand process. The resulting forecasting model will be examined for sensitivity of model parameter estimates to demand observations and for accuracy through simulated application.

Literature Review

Bayesian forecasting techniques have been applied in a variety of situations. Some of the more interesting applications are reviewed.

Cohen [1] develops a Bayesian forecasting model for a demand process with unknown and constant mean and with known and constant variance. Three cases are considered: (1) normally distributed prior and sampling distributions, (2) gamma prior and Poisson sampling distributions, and (3) distribution-free parameter estimation under the criterion of minimizing the variance of the forecast errors.

Chang and Fyffe [2,3] apply linear filtering to the seasonal style goods forecasting problem assuming that the fraction of total demand sold in each period of the season is known with certainty.

Regression analyses are used to derive initial estimates of seasonal demand and the associated error variance, although the forecasting procedure will admit subjective estimates. The model is not applicable when the fractions of total sales in each period are uncertain.

Montgomery [4] derives results analogous to Cohen's for a time series with linear trend for both the normal and the distribution-free cases. Demand observations are utilized in two least squares estimates of the linear model's parameters.

In a production scheduling context, Murray and Silver [5] develop a Bayesian model for forecast revision in a seasonal style goods environment. A Bernoulli process is assumed where the probability of selling an item is constant but unknown and the size of the demand population is known.

Hertz and Shaffir's [6] forecasting technique for style goods employs prior knowledge by assuming cumulative fraction of total sales in each period follows the cumulative normal distribution. This knowledge is used with observed cumulative sales-to-date in the line-ratio forecasting technique to yield a revised estimate of total sales. The revised estimate is obtained by dividing observed cumulative sales-to-date by the applicable cumulative fraction for the period. The procedure is not of value unless the seasonal pattern exhibited by the cumulative fractions happens to follow the cumulative normal distribution. Because of notational errors, the logic of the model is difficult to follow and remains questionable.

None of the forecasting procedures described has solved the general seasonal style goods forecasting problem where the estimated fraction of total sales in a period may be revised at points within the season.

CHAPTER II

THE FORECASTING MODEL

A style good's season is assumed to consist of n periods. Based on subjective and historical data, management estimates the total demand for the season and the expected cumulative fraction of total demand to be realized by the end of each period. Estimates of demand for each of the n periods may then be computed. As observations on the process become available, these period estimates are revised. Current estimates of future periods' demands will be referred to as forecasts.

Basic Concepts

The development of the forecasting model follows from several basic concepts. Let D_i represent the random variable for demand in period i. Then D_i consists of an underlying "true" process ξ_i plus random noise ϵ_i, i.e.,

$$D_i = \xi_i + \epsilon_i \quad , \qquad \text{for } i = 1,2,\ldots,n.$$

The noise ϵ_i is assumed to be normally distributed with zero mean and variance σ_i^2; in addition, the noise is not correlated. Pertinent relations are now summarized:

$$E(D_i) = \xi_i \quad ,$$

$$VAR(D_i) = \sigma_i^2 \quad ,$$

$$\varepsilon_i \sim N(0,\sigma_i^{\,2}) \quad ,$$

$$E(\varepsilon_i \varepsilon_j) = 0, \quad \text{for } i \neq j \ ,$$

where E and VAR denote expectation and variance, respectively.

The problem is to predict D_i at the end of each period j (j less than i). The best predictor of D_i, in the sense of minimum mean square error, is an estimator of the unknown mean ξ_i [2;p.53]. Therefore, an estimator of ξ_i, made at the end of period j (j less than i), is sought. This estimate is designated as $\hat{\xi}_{i,j}$, and it will be the forecast of demand in period i made at the end of period j, or $F_{i,j}$.

Cumulative Demand Process

The remaining development of the model will consider the cumulative demand process; that is, the accumulated demand through period i, denoted as X_i, is of interest:

$$X_i = \sum_{k=1}^{i} D_k \ , \quad \text{for } i = 1,2,\ldots,n.$$

The following relations hold, recalling the assumption of uncorrelated noise:

$$E(X_i) = \sum_{k=1}^{i} \xi_k \equiv \theta_i \ , \quad \text{for } i = 1,2,\ldots,n,$$

$$VAR(X_i) = \sum_{k=1}^{i} \sigma_k^{\,2} \qquad .$$

Having defined θ_i as the expected value of X_i, let ρ_i represent the ratio of θ_i to θ_n (the expected total demand for the season), i.e.,

$$\rho_i = \theta_i/\theta_n \quad .$$

Then, it follows that

$$\rho_o = 0, \ \rho_n = 1, \ 0 \leqq \rho_i \leqq 1, \ \rho_i \leqq \rho_{i+1} \quad .$$

One may interpret ρ_i as the cumulative fraction of total demand realized through period i, and ρ_i will be called the "seasonal factor" for period i. A seasonal pattern results when the n seasonal factors are placed in a continuous framework. Now, an expression for ξ_i is available:

$$\xi_i = (\rho_i - \rho_{i-1})\theta_n,$$

$$= \theta_i - \theta_{i-1} \ .$$

Consequently, there are two sets of parameters that represent the mean ξ_i:

$$(\rho_{i-1}, \rho_i, \theta_n) \ ,$$

or equivalently,

$$(\theta_{i-1}, \theta_i) \quad .$$

At the end of period i-1, forecasts of the demand in periods (i, i+1, ..., n) are needed. These forecasts require estimates of the following parameters:

$$(\rho_{i-1}, \rho_i, \ldots, \rho_{n-1}, \theta_n) \quad ,$$

or

$$(\theta_{i-1}, \theta_i, \ldots, \theta_{n-1}, \theta_n) \quad .$$

Initial Estimation

At time zero, management supplies the means, the variances, and the forms of the distributions for seasonal demand and for the n seasonal factors. The information provided is summarized below.

Prior Distribution	Mean	Variance
seasonal demand, $f_o(\theta_n)$	$\bar{\theta}_{n,o}$	$v_{n,o}$
seasonal factors, $b_{i,o}(\rho_i)$	$\bar{\rho}_{i,o}$	$s_{i,o}$

The prior distributions of cumulative demand through period i, (θ_i), are now constructed. The assumption of independence between each seasonal factor and the seasonal demand is made, that is, the joint distribution of ρ_i and θ_n is factorable into the individual density functions $b_i(\rho_i)$ and $f(\theta_n)$, respectively. Making the substitution of θ_i/ρ_i for θ_n, the prior distribution for θ_i may be constructed [7;p.99]:

$$h_{i,o}(\theta_i) = \int_o^1 f_o\left(\frac{\theta_i}{\rho_i}\right) b_{i,o}(\rho_i)\left(\frac{1}{\rho_i}\right) d\rho_i \quad , \quad \text{for } i = 1,2,\ldots,n-1,$$

with mean $\bar{\theta}_{i,o}$ and variance $v_{i,o}$. An estimate, designated by a circumflex, of θ_i at time zero is the mean, or

$$\hat{\theta}_{i,o} = \bar{\theta}_{i,o} \quad .$$

The expressions for the mean and variance of $h_{i,o}(\theta_i)$ are the following, recalling the assumption of independence of ρ_i and θ_n:

$$\bar{\theta}_{i,o} = \bar{\rho}_{i,o}\bar{\theta}_{n,o} \quad ,$$

$$v_{i,o} = E(\rho_{i,o}^{2}\theta_{n,o}^{2}) - (\bar{\rho}_{i,o}\bar{\theta}_{n,o})^2 \quad .$$

The variance expression may be rewritten as

$$v_{i,o} = s_{i,o}v_{n,o} + s_{i,o}\bar{\theta}_{n,o}^{2} + v_{n,o}\bar{\rho}_{i,o}^{2} \quad ,$$

[8;p.6]. The equivalence of the two expressions is verified upon expansion of the latter (assuming ρ_i^2 and θ_n^2 are independent as well).

As noted earlier, estimates of the parameters $(\theta_{i-1}, \theta_i, \ldots, \theta_n)$ are needed at the beginning of period i for forecasting purposes. At time zero, the estimators $(\theta_0 = 0, \hat{\theta}_{1,0} = \bar{\theta}_{1,0}, \ldots, \hat{\theta}_{n,0} = \bar{\theta}_{n,0})$ are computed, and the forecasts are made:

$$F_{1,0} = \bar{\theta}_{1,0} - \theta_0 = \bar{\theta}_{1,0} \quad ,$$

$$F_{2,0} = \bar{\theta}_{2,0} - \bar{\theta}_{1,0} \quad ,$$

$$\vdots$$

$$F_{n,0} = \bar{\theta}_{n,0} - \bar{\theta}_{n-1,0} \quad .$$

Posterior Estimation

Observations of the demand process are made at the end of each period. The cumulative demand observation is denoted by x_i, and the sampling distribution is $g_i(x_i | \theta_i)$. The problem is to revise the estimate of θ_i in light of the observation x_i. A Bayesian approach is taken. For an expected squared error loss function, it is well known that the Bayesian estimator of θ_i is the mean of the posterior density $h_{i,i}(\theta_i | x_i)$ [9;p.59]. The posterior density is found by Bayes' Theorem noting that prior information is of the form $h_{i,i-1}(\theta_i)$ immediately preceding revision:

$$h_{i,i}(\theta_i | x_i) = \frac{h_{i,i-1}(\theta_i) g_i(x_i | \theta_i)}{\int h_{i,i-1}(\theta_i) g_i(x_i | \theta_i) d\theta_i} \quad ;$$

$$\text{mean: } \bar{\theta}_{i,i}, \qquad \text{variance: } v_{i,i} \quad .$$

The posterior density of θ_n upon observing x_n is formed in the same manner.

If, for illustration, a normal process is assumed with $VAR(X_i)$ known, the posterior mean and variance are easily found [10;p.441]:

$$\bar{\theta}_{i,i} = \frac{v_{i,i-1} x_i + VAR(X_i) \bar{\theta}_{i,i-1}}{v_{i,i-1} + VAR(X_i)} \quad ,$$

$$v_{i,i} = \frac{v_{i,i-1} VAR(X_i)}{v_{i,i-1} + VAR(X_i)} \quad .$$

As long as the prior and the sampling distributions are conjugate, posterior estimates are readily obtained [11;pp.53-58]. At this point, revision of the remaining densities must be performed.

Seasonal Factors Known

If the seasonal factors are known at the beginning of the season, the problem is considerably reduced in complexity. At time zero, the prior distributions, means, and variances are computed from the following relations:

$$h_{i,o}(\theta_i) = \frac{1}{\rho_i} f_o\left(\frac{\theta_i}{\rho_i}\right) \quad , \qquad (2\text{-}1)$$

$$\bar{\theta}_{i,o} = \rho_i \bar{\theta}_{n,o} \ ,$$

$$v_{i,o} = \rho_i^2 v_{n,o} \quad .$$

The posterior distribution of $h_i(\theta_i)$ upon observing x_i is found using Bayes' Theorem, and the remaining distributions' means and variances are easily revised, i.e.,

$$\bar{\theta}_{n,i} = \bar{\theta}_{i,i}/\rho_i \quad , \qquad (2\text{-}2)$$

$$v_{n,i} = v_{i,i}/\rho_i^2 \ ,$$

$$\bar{\theta}_{j,i} = \rho_j \bar{\theta}_{n,i} \ , \ j \neq i, \ j \neq n \quad ,$$

$$v_{j,i} = \rho_j^2 v_{n,i} \quad .$$

The revised distributions with index greater than i may still be interpreted as prior knowledge.

In general, the forecasts at the end of period i may be written as:

$$F_{i+1,i} = \bar{\theta}_{i+1,i} - \bar{\theta}_{i,i} \quad ,$$

$$F_{i+2,i} = \bar{\theta}_{i+2,i} - \bar{\theta}_{i+1,i} \quad ,$$

$$\vdots$$

$$F_{n,i} = \bar{\theta}_{n,i} - \bar{\theta}_{n-1,i} \quad .$$

Another solution to the seasonal style goods forecasting problem under the assumption of known seasonal factors was developed by Chang as mentioned in the literature survey. Comparison of the present model with Chang's will be made in Chapter III.

Seasonal Factors Unknown

A general solution of the forecasting problem when the seasonal factors are unknown is not available from preceding relationships. Reconsider the equations developed earlier with the time of estimation modified for posterior use:

$$\bar{\theta}_{i,i} = \bar{\rho}_{i,i}\bar{\theta}_{n,i} \quad ,$$

$$v_{i,i} = s_{i,i}v_{n,i} + s_{i,i}\bar{\theta}^2_{n,i} + v_{n,i}\bar{\rho}^2_{i,i} \quad .$$

After observing x_i, the values of $\bar{\theta}_{i,i}$ and $v_{i,i}$ are found through Bayes' Theorem. Four variables remain in the two equations, preventing general revisions of both total demand for the season and the seasonal factor; restrictions must be imposed on two of these variables [8;p.7].

In practice specifying values for two of the variables may be less restrictive than first thought. For example, management might be quite confident in the initial estimate of the demand for the season, but may be unsure of the seasonal pattern. Posterior estimates of the mean $\bar{\rho}_i$ and the variance s_i of the seasonal factors are sought. These estimates may be obtained if the mean $\bar{\theta}_n$ and the variance v_n are specified. Suppose that past experience with a style line has indicated that the variance v_n decreases as a known function of time. Further, management is satisfied that the estimate $\bar{\theta}_{n,o}$ is representative of the current season, and is willing to revise this estimate subjectively during the season if marketing reports indicate the need for such a change. As a result, posterior estimates of the mean and variance of the seasonal factor for a period may be computed, enabling management to revise the remaining seasonal factors on a more objective basis.

In another case, the variances v_n and s_i might be assumed to decrease at known rates, and posterior estimates of the means $\bar{\theta}_n$ and $\bar{\rho}_i$ could be obtained simultaneously from the two equations.

With four variables, there are six possible ways of restricting two of the values so that management can model practically any situation. Flexibility of the procedure is further extended as combinations of two or more of the six variations could be utilized at different times within a season. A combination of two variations will be demonstrated in the next chapter. The general approach of the forecasting procedure is now summarized.

The observation x_i has been made and $\bar{\theta}_{i,i}$ and $v_{i,i}$ have been found through Bayes' Theorem; posterior estimates of two variables are sought, and the values of the remaining two variables have been specified. Posterior estimates are obtained through simultaneous solution of the following equations:

$$\bar{\theta}_{i,i} = \bar{\rho}_{i,i}\bar{\theta}_{n,i} \quad ,$$

$$v_{i,i} = s_{i,i}v_{n,i} + s_{i,i}\bar{\theta}^2_{n,i} + v_{n,i}\bar{\rho}^2_{i,i} \; .$$

Estimates of the demands in other periods are revised:

$$\bar{\theta}_{j,i} = \bar{\rho}_{j,i}\bar{\theta}_{n,i} \quad , \text{ for } j \neq i \quad ,$$

$$v_{j,i} = s_{j,i}v_{n,i} + s_{j,i}\bar{\theta}^2_{n,i} + v_{n,i}\bar{\rho}^2_{j,i} \quad .$$

Forecasts are made:

$$F_{j,i} = \bar{\theta}_{j,i} - \bar{\theta}_{j-1,i} \quad , \quad \text{for } j = i + 1, \ldots, n.$$

The above procedure has provided a restricted solution to the general seasonal style goods forecasting problem.

Comments

Solution of the general seasonal style goods forecasting problem is not available from relationships developed herein. However, if seasonal factors are known, the problem is solvable, and a solution has been presented. When seasonal factors are unknown, solution to the forecasting problem may be found without being unduly restrictive in many cases.

CHAPTER III

MODEL EVALUATION

Sensitivity, accuracy, and precision of the forecasting model will be examined at this point. Sensitivity will be determined by analyzing the effects of limiting values of noise variance and prior variance on the posterior estimate of a period's demand. Also, the reduction of the variance of the estimate upon demand observation is demonstrated. Afterwards, accuracy and precision of the model will be considered concurrently through simulated applications. A comparison with Chang's model and the line-ratio technique (see literature review) assuming known seasonal factors will be made; later, the model is evaluated for the unknown seasonal factors case.

Sensitivity

The posterior estimate of a period's demand directly reflects the variability associated with the noise and with the prior knowledge. The effects of limiting values of these two variances is of particular interest. Assuming normality, the posterior mean and variance were shown to be:

$$\bar{\theta}_{i,i} = \frac{v_{i,i-1}x_i + VAR(X_i)\bar{\theta}_{i,i-1}}{v_{i,i-1} + VAR(X_i)} \quad ,$$

$$v_{i,i} = \frac{v_{i,i-1}VAR(X_i)}{v_{i,i-1} + VAR(X_i)} \quad .$$

If the variance associated with the prior knowledge, $v_{i,i-1}$, approaches zero, the posterior estimate of the mean will remain the same as the prior estimate. Similarly, if the variance of the noise, $VAR(X_i)$, approaches zero, the posterior estimate of the mean will become the value of the observation. The posterior variance is zero in either case. These results are intuitively clear and may be found by taking the limits of the posterior expressions, i.e.,

$$\lim_{v_{i,i-1}\to 0} \bar{\theta}_{i,i} = \bar{\theta}_{i,i-1} \quad ,$$

$$\lim_{v_{i,i-1}\to 0} v_{i,i} = 0 \quad ,$$

$$\lim_{VAR(X_i)\to 0} \bar{\theta}_{i,i} = x_i \quad ,$$

$$\lim_{VAR(X_i)\to 0} v_{i,i} = 0 \quad .$$

When the prior variance approaches infinity, the following results are apparent:

$$\lim_{v_{i,i-1}\to \infty} \bar{\theta}_{i,i} = x_i \quad ,$$

$$\lim_{v_{i,i-1}\to \infty} v_{i,i} = VAR(X_i) \quad .$$

And, finally, as the noise variance approaches infinity, the limits become:

$$\lim_{VAR(X_i)\to\infty} \bar{\theta}_{i,i} = \bar{\theta}_{i,i-1} \quad ,$$

$$\lim_{VAR(X_i)\to\infty} v_{i,i} = v_{i,i-1} \quad .$$

The fact that the posterior variance is less than the prior variance is an important result that is easily demonstrated. The variance expression may be rewritten as

$$v_{i,i} = \frac{v_{i,i-1}}{\frac{v_{i,i-1}}{VAR(X_i)} + 1} \quad .$$

Since the denominator is obviously greater than one (except in the limiting cases), the posterior variance is less than the prior variance.

The above results indicate the sensitivity of the posterior estimates to the variances of the noise and the prior knowledge, and also demonstrate the reduced variability of the posterior estimate.

Forecasting with Known Seasonal Factors

Assuming that the seasonal factors are known, the present model, hereafter referred to as the "Bayesian" model, will be compared to Chang's model and the line-ratio technique for accuracy and precision. First, it will be instructive to describe the general approach of Chang's procedure.

Defining a residual as the difference between the observed value of demand for a period and the forecast made one period earlier, Chang periodically revises the initial estimate of total demand based on a weighted average of the current residual and the accumulated residuals to date. Since the seasonal factors are known with certainty, each period's demand is also easily revised. The residuals are assumed to have a multivariate normal distribution. Chang then shows that linear operations on the residuals are sufficient for revision of the season's demand estimate. Using a mean squared error loss function (modified to allow only linear estimates), the expressions for the weights in the averaging process are derived in terms of the noise variance and the seasonal demand estimate's variance. The procedure is called "linear filtering."

The line-ratio technique obtains a posterior estimate of total demand by dividing cumulative sales-to-date by the appropriate seasonal factor, i.e.,

$$\hat{x}_{n,i} = x_i/\rho_i \quad .$$

Note that the procedure attempts to estimate x_n directly rather than the mean $\bar{\theta}_n$.

The data to be used in the evaluation are from R. G. Brown's Smoothing, Forecasting, and Prediction [12;p.432]. Brown suggests a cosine function without trend to represent the demand process of Warmdot filters. Although the data are from a continuous demand process, September through June will represent a season of ten periods for present purposes. Forecasts of the demand in the 1961-62 season will be made.

Since historical data are available, the seasonal factors and the noise variances will be computed statistically by the methods presented in Chang's thesis [2;pp.95-99]. Further, initial estimates of the mean and variance of seasonal demand will be determined from a sample of past data (although the models will admit subjective data if available). Chang's statistics and the values obtained from the referenced data are found in the Appendix. Assuming a normal process with known variance, the Bayesian estimates are computed from equation sets (2-1) and (2-2).

The estimates that were generated by the three models are presented in Tables 1, 2, 3. Both the Bayesian model and Chang's model periodically reduce the variance of the total demand estimate, but the Bayesian model performs this function more rapidly. As a result, Chang's model weighs current observations more heavily than does the Bayesian model; the reverse is true for prior knowledge. In period three of Table 2, it appears that Chang's model behaves inconsistently. The prior estimate of demand is $\bar{\theta}_{3,2} = 466$, and the observation x_3 equals 463; one would suspect that the posterior estimate would be revised downwardly, but it is not. Recalling that Chang's model is not based on a cumulative demand process (as is the Bayesian model), one dismisses the inconsistency as the upward revision is consistent with the residual for the period.

Accuracy and precision of the three models were measured by the average error of the forecast (A.E.) and the mean absolute deviation (M.A.D.) respectively. Lead time is defined as the number of periods that expire between the time in which a forecast for a period is made and the time in which the demand observation for that period occurs. Average error simply averages the residuals between the forecasts and the observations for the lead time under consideration. Mean absolute

Table 1. Estimates Generated by Bayesian Model

i	$\bar{\theta}_{i,i-1}$	$v_{i,i-1}$	x_i	$\bar{\theta}_{i,i}$	$v_{i,i}$	$\bar{\theta}_{10,i}$	$v_{10,i}$
0						1462	7279
1	132	59	127	131	56	1459	6885
2	292	275	267	288	232	1440	5808
3	469	617	463	468	486	1436	4570
4	663	975	675	666	716	1442	3354
5	841	1140	880	852	831	1461	2444
6	1003	1154	1005	1004	858	1461	1818
7	1147	1121	1127	1143	880	1456	1428
8	1275	1096	1226	1265	875	1444	1141
9	1364	1016	1360	1363	839	1444	942
10	1444	942	1512	1453	810	1453	810

Table 2. Estimates Generated by Chang's Model

i	$\bar{\theta}_{i,i-1}$	$v_{i,i-1}$	x_i	$\bar{\theta}_{i,i}$	$v_{i,i}$	$\bar{\theta}_{10,i}$	$v_{10,i}$
0						1462	7279
1	132	59	127	131	56	1459	6885
2	292	275	267	286	233	1430	5823
3	466	619	463	471	554	1444	5214
4	667	1113	675	677	903	1465	4229
5	854	1437	880	873	1231	1498	3621
6	1029	1709	1005	1005	1506	1463	3192
7	1149	1967	1127	1142	1890	1454	3066
8	1274	2353	1226	1244	2132	1420	2778
9	1341	2476	1360	1354	2409	1435	2704
10	1435	2704	1512	1446	2680	1446	2680

Table 3. Estimates Generated by Line-Ratio Technique

i	x_i	ρ_i	$x_{10,i}$
0			1462
1	127	.090	1411
2	267	.200	1335
3	463	.326	1420
4	675	.462	1461
5	880	.583	1509
6	1005	.687	1463
7	1127	.785	1436
8	1226	.876	1400
9	1360	.944	1441
10	1512	1.000	1512

deviation averages the absolute values of the residuals for a given lead time. The average error and the mean absolute deviation are computed for one, two, and six period lead times. Let d_i be a noncumulative period observation. See Tables 4, 5, 6.

With regard to the two measures of effectiveness, the Bayesian model and Chang's model yield almost identical results for the given data. Both models are more effective than the line-ratio technique in every case but one (mean absolute deviation for a six period lead time). The difference between the Bayesian model, assuming a normal process, and Chang's model is that the Bayesian model is designed for a cumulative demand process.

Forecasting with Unknown Seasonal Factors

The subsequent computations were made to demonstrate the approach to the general seasonal style goods forecasting problem as described in Chapter II. In the general problem, neither the seasonal factors nor the total demand is known with certainty, but, in most cases of practical interest, a model of sufficient accuracy for forecasting purposes may still be formulated. As results are of present concern, the reader is referred to Chapter II for the underlying development of this section.

As previously indicated, in some situations it may be desirable to use various combinations of specified and unspecified variables at different times in the season in order to obtain posterior estimates from the two equations below:

Table 4. A.E. and M.A.D. for a One Period Lead Time

	Bayesian Model			Chang's Model			Line-Ratio		
i	F_i	d_i	F_i-d_i	F_i	d_i	F_i-d_i	F_i	d_i	F_i-d_i
1	132	127	5	132	127	5	132	127	5
2	161	140	21	161	140	21	155	140	15
3	181	196	-15	180	196	-16	168	196	-28
4	195	212	-17	196	212	-16	193	212	-19
5	175	205	-30	177	205	-28	177	205	-28
6	152	125	27	156	125	31	157	125	32
7	143	122	21	143	122	21	143	122	21
8	132	99	33	132	99	33	131	99	32
9	98	134	-36	97	134	-37	95	134	-39
10	81	152	-71	80	152	-72	81	152	-71
	AE = -6.2 MAD = 27.6			AE = -5.8 MAD = 28.0			AE = -8.0 MAD = 29.0		

Table 5. A.E. and M.A.D. for a Two Period Lead Time

	Bayesian Model			Chang's Model			Line-Ratio		
i	F_i	d_i	F_i-d_i	F_i	d_i	F_i-d_i	F_i	d_i	F_i-d_i
2	161	140	21	161	140	21	161	140	21
3	184	196	-12	184	196	-12	178	196	-18
4	196	212	-16	195	212	-17	182	212	-30
5	174	205	-31	175	205	-30	172	205	-33
6	150	125	25	152	125	27	152	125	27
7	143	122	21	147	122	25	148	122	26
8	133	99	34	133	99	34	133	99	34
9	99	134	-35	99	134	-35	98	134	-36
10	81	152	-71	80	152	-72	78	152	-74
AE = -7.1		MAD = 29.6		AE = -6.6		MAD= 30.3	AE = -9.2		MAD = 33.2

Table 6. A.E. and M.A.D. for a Six Period Lead Time

	Bayesian Model			Chang's Model			Line-Ratio		
i	F_i	d_i	F_i-d_i	F_i	d_i	F_i-d_i	F_i	d_i	F_i-d_i
6	152	125	27	152	125	27	152	125	27
7	143	122	21	143	122	21	138	122	16
8	131	99	32	130	99	31	121	99	22
9	98	134	-36	98	134	-36	97	134	-37
10	81	152	-71	82	152	-70	82	152	-70
	AE = -5.4 MAD = 37.4			AE = -5.4 MAD = 37.0			AE = -8.4 MAD = 34.4		

$$\bar{\theta}_{i,i} = \bar{\rho}_{i,i}\bar{\theta}_{n,i} \quad , \tag{3-1}$$

$$v_{i,i} = s_{i,i}v_{n,i} + s_{i,i}\bar{\theta}_{n,i}^{2} + v_{n,i}\bar{\rho}_{i,i}^{2} \quad . \tag{3-2}$$

For example, past experience with a style line may indicate that an individual style will follow one of three possible patterns, perhaps as illustrated below.

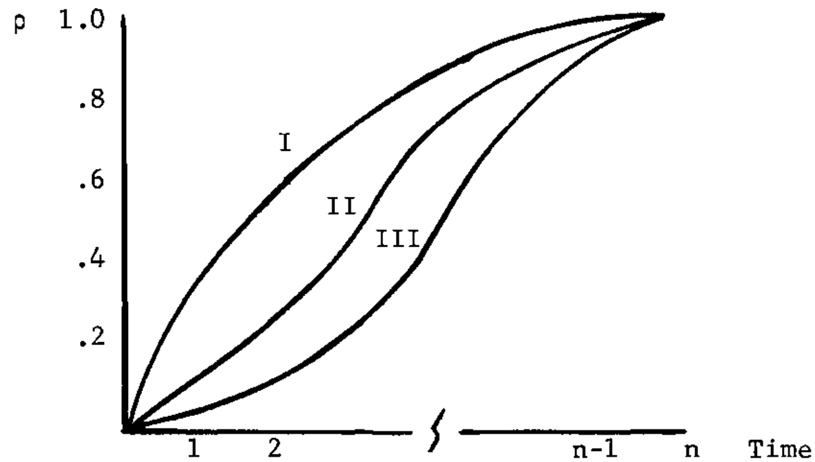

Figure 1. Seasonal Factors Patterns

As assumed earlier, the pattern followed is independent of total demand.

At the beginning of the season, management does not know the correct pattern; therefore, all patterns are considered possible for the first few periods of the season. Seasonal factors estimates $\bar{\rho}_{i,o}$ are thereby specified. Management will obtain posterior estimates assuming each pattern in turn, and after a few observations the correct pattern may be chosen.

The variances $s_{i,0}$ may be approximated by the following expression:

$$s_{i,0} = \frac{\sum_{j=1}^{k} \left(\frac{x_{i,j}}{x_{n,j}} - \bar{\rho}_{i,0}\right)^2}{k - 1} , \quad i = 1, 2, \ldots, n-1,$$

where the index j specifies data values of styles that followed the pattern under consideration in past seasons.

As posterior estimates are found for all patterns initially, management can compare the results with the three possible patterns. The posterior estimates of the seasonal factors will not follow the exact values of the projected patterns since two of the patterns are incorrect and since an individual style will deviate about the correct pattern. As a result, "control" limits about the seasonal factors estimates are needed.

An estimate of the variation of the posterior estimate about the prior estimate is given:

$$\Phi_{i,i}^2 = \frac{\sum_{j=1}^{k} (\bar{\rho}_{i,i,j} - \bar{\rho}_{i,0})^2}{k - 1}$$

where j is interpreted as before. Control limits can now be constructed for the seasonal factors. For example, if "three-sigma" limits are desired, a seasonal pattern would not be eliminated from consideration as long as the posterior estimates $\bar{\rho}_{i,i}$ were within three standard deviations of the prior mean, i.e.,

$$\bar{\rho}_{i,0} - 3\Phi_{i,i} \leq \bar{\rho}_{i,i} \leq \bar{\rho}_{i,0} + 3\Phi_{i,i} \quad .$$

Symmetrical limits are not required. The lower and upper control limits will henceforth be designated as LCL and UCL respectively. Control limits may also be established subjectively.

Until the correct pattern is found, posterior estimates of the mean $\bar{\rho}_i$ and the variance s_i of the seasonal factors are desired. Certain assumptions must be made with regard to the mean $\bar{\theta}_n$ and the variance v_n of total demand to allow solution of the two simultaneous equations (3-1) and (3-2). In this case, holding the initial estimate of the mean constant and assuming that the variance decreases as a known function of time is assumed to be reasonable. After the incorrect patterns are eliminated, the seasonal factors $\bar{\rho}_i$ may be held constant and the variance s_i can be assumed to decrease at a known rate; posterior estimates of θ_n and v_n may then be obtained. This procedure allows determination of the correct seasonal pattern early in the season, as well as permits timely revision of the total demand estimate so that obsolescence costs may be avoided. The procedure is now summarized.

The observation x_i has been made, and $\bar{\theta}_{i,i}$ and $v_{i,i}$ have been found through Bayes' Theorem. Until the incorrect patterns are eliminated, the value of $v_{n,i}$ is determined from the applicable relation evaluated at period i; the initial estimate of θ_n is utilized. The following equations are solved simultaneously for the estimates $\bar{\rho}_{i,i}$ and $s_{i,i}$:

$$\bar{\theta}_{i,i} = \bar{\rho}_{i,i}\bar{\theta}_{n,0} \quad , \tag{3-3}$$

$$v_{i,i} = s_{i,i}v_{n,i} + s_{i,i}\bar{\theta}_{n,0}^{2} + v_{n,i}\bar{\rho}_{i,i}^{2} \quad . \tag{3-4}$$

Prior and posterior estimates are computed for each of the three patterns initially. The estimates $\bar{\rho}_{i,i}$ are tested for conformance with the corresponding control limits; incorrect patterns are eliminated. If more than one pattern remains, revised estimates of demand in future periods are determined in the following manner where $s_{j,i}$ (j greater than i) is obtained from an appropriate relation:

$$\bar{\theta}_{j,i} = \bar{\rho}_{j,0}\bar{\theta}_{n,0}, \qquad \text{for } j > i \quad , \tag{3-5}$$

$$v_{j,i} = s_{j,i}v_{n,i} + s_{j,i}\bar{\theta}_{n,0}^{2} + v_{n,i}\bar{\rho}_{j,0}^{2} \quad . \tag{3-6}$$

Note that seasonal factors estimates $\bar{\rho}_{j,0}$ are not altered from their initial values as posterior estimates $\bar{\rho}_{i,i}$ (i less than j) have indicated that the remaining patterns are the best choices among the three alternatives. Forecasts are made at this point by choosing one of the patterns subjectively:

$$F_{j,i} = \bar{\theta}_{j,i} - \bar{\theta}_{j-1,i} \quad , \quad \text{for } j > i \ . \tag{3-7}$$

After each observation, the procedure is performed until only one pattern remains. When the correct pattern is found, revisions of future period demands are made as above for one more period. From that point forward, posterior estimates of θ_n and v_n are computed from the two simultaneous equations using the seasonal factor $\bar{\rho}_{i,0}$ of the correct pattern and the value of s_i as determined by the applicable relation evaluated in period i, i.e.,

$$\bar{\theta}_{i,i} = \bar{\rho}_{i,0}\bar{\theta}_{n,i} \quad , \tag{3-8}$$

$$v_{i,i} = s_{i,i}v_{n,i} + s_{i,i}\bar{\theta}_{n,i}^{2} + v_{n,i}\bar{\rho}_{i,0}^{2} \quad . \tag{3-9}$$

Revised estimates of demand in future periods are made:

$$\bar{\theta}_{j,i} = \bar{\rho}_{j,0}\bar{\theta}_{n,i} \, , \qquad \text{for } j > i \quad , \tag{3-10}$$

$$v_{j,i} = s_{j,i}v_{n,i} + s_{j,i}\bar{\theta}_{n,i}^{2} + v_{n,i}\bar{\rho}_{j,0}^{2} \, . \tag{3-11}$$

Demand is forecasted using equation (3-7) above.

The following computations demonstrate the approach. Without impairing the illustrative purpose of this section, initial estimates and the expressions for the variances of v_n and s_i have been fabricated. A normal demand process with known variance is assumed.

Suppose that management subjectively estimates the demand for the season and the variance of the estimate as follows:

$$\bar{\theta}_{5,0} = 1365 \quad , \qquad v_{5,0} = 1600 \quad .$$

The demand observations to be predicted and the noise variances are given.

i	x_i	$VAR(X_i)$
1	115	1101
2	213	1623
3	564	2250
4	1012	2917
5	1341	3649

Table 7 presents the three sets of seasonal factors estimates, their variances, and the confidence limits for posterior analysis. When posterior estimates are not to be obtained for v_5 and s_i through equations (3-1) and (3-2), the variances may be evaluated in period j by the following relations:

$$v_{5,j} = (1 - (.01j)^2)\ v_{5,0} \quad , \tag{3-12}$$

$$s_{i,j} = (1 - .2j)\ s_{i,0} \quad . \tag{3-13}$$

At time zero, management computes prior estimates of $\bar{\theta}_{1,0}$ and $v_{1,0}$ for the three patterns using equations (3-5) and (3-6).

Pattern	$\bar{\theta}_{1,0}$	$v_{1,0}$
I	710	619
II	109	197
III	382	312

The observation x_1 is made, and $v_{5,1}$ is computed from equation (3-12). Posterior estimates are found through Bayes' Theorem and equations (3-3) and (3-4).

Pattern	$\bar{\theta}_{1,1}$	$v_{1,1}$	$\bar{\rho}_{1,1}$	$s_{1,1}$
I	496	396	.363	.000099
II	110	167	.081	.000084
III	323	243	.237	.000082

Table 7. Estimates and Control Limits

	Pattern I				Pattern II				Pattern III			
i	$\bar{\rho}_{i,0}$	$s_{i,0}$	LCL	UCL	$\bar{\rho}_{i,0}$	$s_{i,0}$	LCL	UCL	$\bar{\rho}_{i,0}$	$s_{i,0}$	LCL	UCL
1	.52	.0001	.49	.55	.08	.0001	.05	.12	.28	.0001	.23	.31
2	.71	.0003	.65	.75	.16	.0001	.12	.21	.40	.0002	.35	.46
3	.82	.0003	.77	.86	.42	.0002	.37	.48	.61	.0003	.56	.67
4	.94	.0001	.90	.99	.73	.0003	.66	.79	.84	.0002	.79	.88
5	1.00	0	--	--	1.00	0	--	--	1.00	0	--	--

Comparing the estimates $\bar{\rho}_{1,1}$, with the control limits of Table 7, one may eliminate Pattern I. Prior estimates are now computed for $\bar{\theta}_{2,1}$ and $v_{2,1}$ with (3-5), (3-6), and (3-13).

Pattern	$\bar{\theta}_{2,1}$	$v_{2,1}$
II	218	190
III	546	554

Posterior estimates are found as above.

Pattern	$\bar{\theta}_{2,2}$	$v_{2,2}$	$\bar{\rho}_{2,2}$	$s_{2,2}$
II	218	170	.160	.000069
III	461	413	.338	.000120

Pattern III is eliminated. Prior estimates $\bar{\theta}_{3,2}$ and $v_{3,2}$ are computed for Pattern II:

$$\bar{\theta}_{3,2} = 573 \quad , \quad v_{3,2} = 506 \quad .$$

Posterior estimates are now determined with equations (3-8) and (3-9):

$$\bar{\theta}_{3,3} = 572 \quad , \quad v_{3,3} = 413 \quad , \quad \bar{\theta}_{5,3} = 1361 \quad , \quad v_{5,3} = 1501 \quad .$$

Equations (3-10), (3-11), and (3-13) yield values for $\bar{\theta}_{4,3}$ and $v_{4,3}$:

$$\bar{\theta}_{4,3} = 993 \quad , \quad v_{4,3} = 1022 \quad .$$

Upon observing x_4 and x_5, the following two sets of computations are made:

$$\bar{\theta}_{4,4} = 998 \quad , \quad v_{4,4} = 757 \quad , \quad \bar{\theta}_{5,4} = 1368 \quad , \quad v_{5,4} = 1210 \;;$$

and

$$\bar{\theta}_{5,5} = 1361, \quad v_{5,5} = 909 \quad .$$

Forecasts are made with equation (3-7) in each case. When more than one pattern remains, management must subjectively choose a pattern for forecasting purposes. In this example, Pattern II would probably be chosen after x_1 was observed. The forecasts that Pattern II would yield for a one period lead time are given below.

i	F_i	d_i
1	109	115
2	108	98
3	355	351
4	422	448
5	369	329

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The applicability of Bayes' Theorem for forecasting the demand of seasonal style goods has been examined. In the general problem, neither the seasonal factors nor the total demand is known with certainty. The restricted case of known seasonal factors has been solved through linear filtering previously [2,3]. In this paper, Bayesian approaches to the seasonal style goods forecasting problem have been developed for both the unknown and the known seasonal factors cases. Conclusions with regard to the use of Bayes' Theorem as a method for demand forecasting will be drawn first.

Bayes' Theorem can effectively draw upon both prior knowledge and demand observations in revising estimates of model parameters.

When the prior and the sample distributions are conjugate, posterior estimates of the mean and variance of period demand are efficiently obtained through Bayes' Theorem.

The Bayesian approach readily accepts initial estimates that are based on either subjective or objective data or a combination of the two.

The following conclusions pertain to the seasonal style goods problem with known seasonal factors.

With regard to measures of forecast accuracy and precision, the Bayesian model and the Chang model are equally effective, and both models compare favorably to the widely used line-ratio technique.

The Bayesian model is a simple, direct approach requiring near minimal computation time.

Conclusions are now given for the unknown seasonal factors case.

The Bayesian model has provided a solution of the general seasonal style goods forecasting problem conditioned on the ability of management to estimate two of four unknown parameters at the time of posterior revision.

Simplicity is maintained with the increase in generality, and computation time remains negligible.

Recommendations

Another approach to the general problem may be found by extending the results of Montgomery [4]. The demand pattern could be modeled as a time series with constant, trend, and periodic components. The periodic component would consist of an appropriate number of sine and cosine terms. A least squares criterion would be used to obtain statistics for revising prior distributions of model parameters. Adaptive smoothing could be used to simplify computations if minimizing the weighted sum of the square residuals were chosen as the criterion. This approach would offer a general solution to the forecasting problem without restriction and should be investigated.

APPENDIX

Computations for Forecasting with Known Seasonal Factors

In order to compute known seasonal factors and noise variances from a sample of past data, Chang [2,3] proposes the following statistics (modified for use in a cumulative demand process), where the index j specifies data values of styles that have followed the pattern in the past.

$$\rho_i = \frac{\sum_{j=1}^{k} x_{i,j}}{\sum_{j=1}^{k} x_{10,j}}$$

$$\mathrm{VAR}(X_i) = \sum_{j=1}^{k} (x_{i,j} - \rho_i x_{10,j})^2/(k-1)$$

The results obtained for the previously referenced data are given on the following page.

Table 8. Known Seasonal Factors and Noise Variances

i	ρ_i	$VAR(X_i)$
1	.090	1029
2	.200	1486
3	.326	2278
4	.462	2692
5	.583	3061
6	.687	3352
7	.785	4101
8	.876	4346
9	.944	4812
10	1.000	5782

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